Music Processing Analysis

Fourier Analysis I

Exercise

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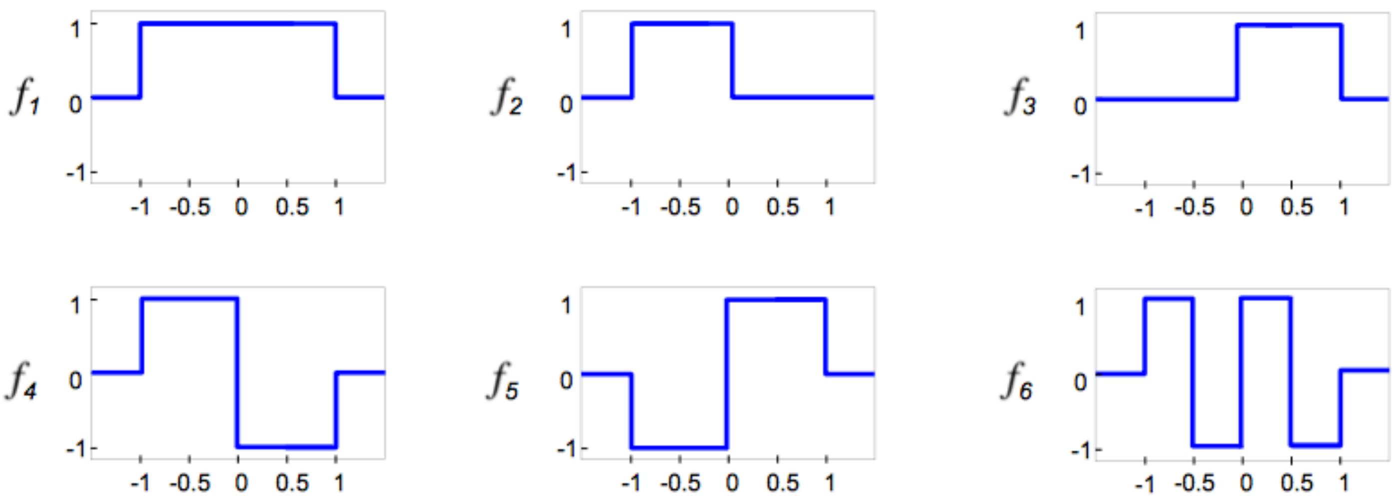
Session Outline
Music Representations

- Homework discussion + demos
- Introduction to Python

Homework

Exercise 2.1

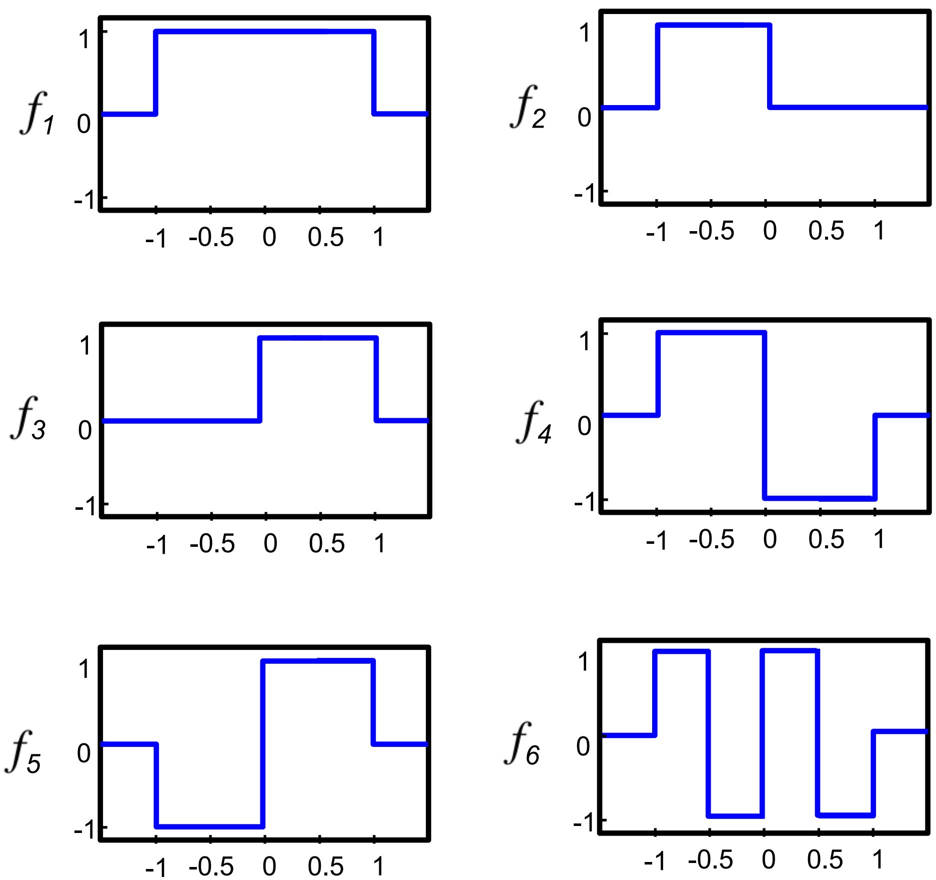
Exercise 2.1. Let $\langle f|g\rangle := \int_{t\in\mathbb{R}} f(t)\cdot g(t)dt$ be the similarity measure for two functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ as defined in (2.3). Consider the following six functions $f_n:\mathbb{R}\to\mathbb{R}$ for $n\in[1:6]$, which are defined to be zero outside the shown interval:



Determine the similarity values $\langle f_n|f_m\rangle$ for all pairs $(n,m)\in[1:6]\times[1:6]$.

Homework

Solution 2.1

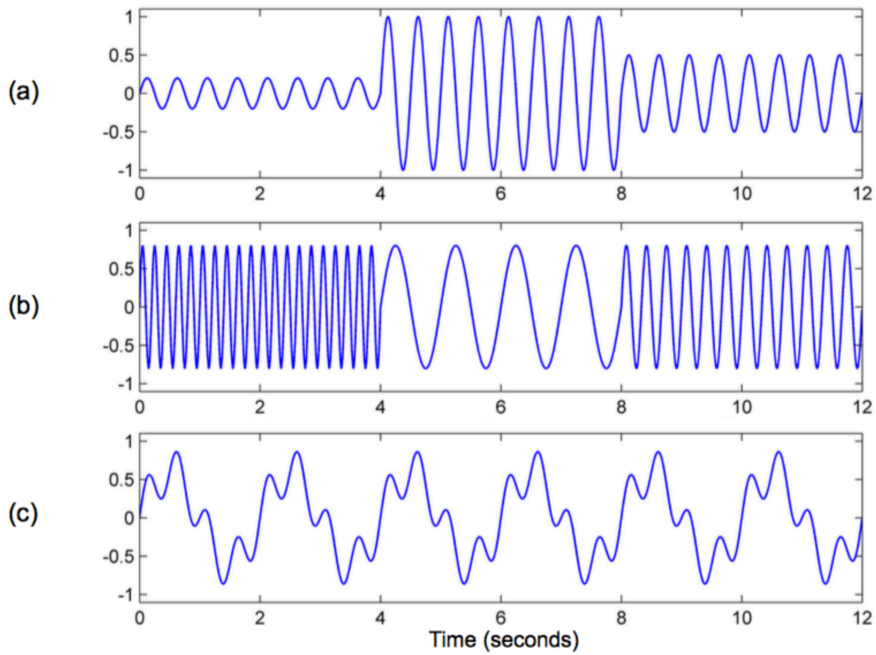


$$\int_{t\in\mathbb{R}} f(t)\cdot g(t)dt. \qquad (2.3)$$

$\langle f_n \mid f_m \rangle$	f_1	f_2	f_3	f_4	f_5	f_6
f_1	2	1	1	0	0	0
f_2	1	1	0	1	-1	0
f_3	1	0	1	-1	1	0
f_4	0	1	-1	2	-2	0
f_5	0	-1	1	-2	2	0
f_6	0	0	0	0	0	2

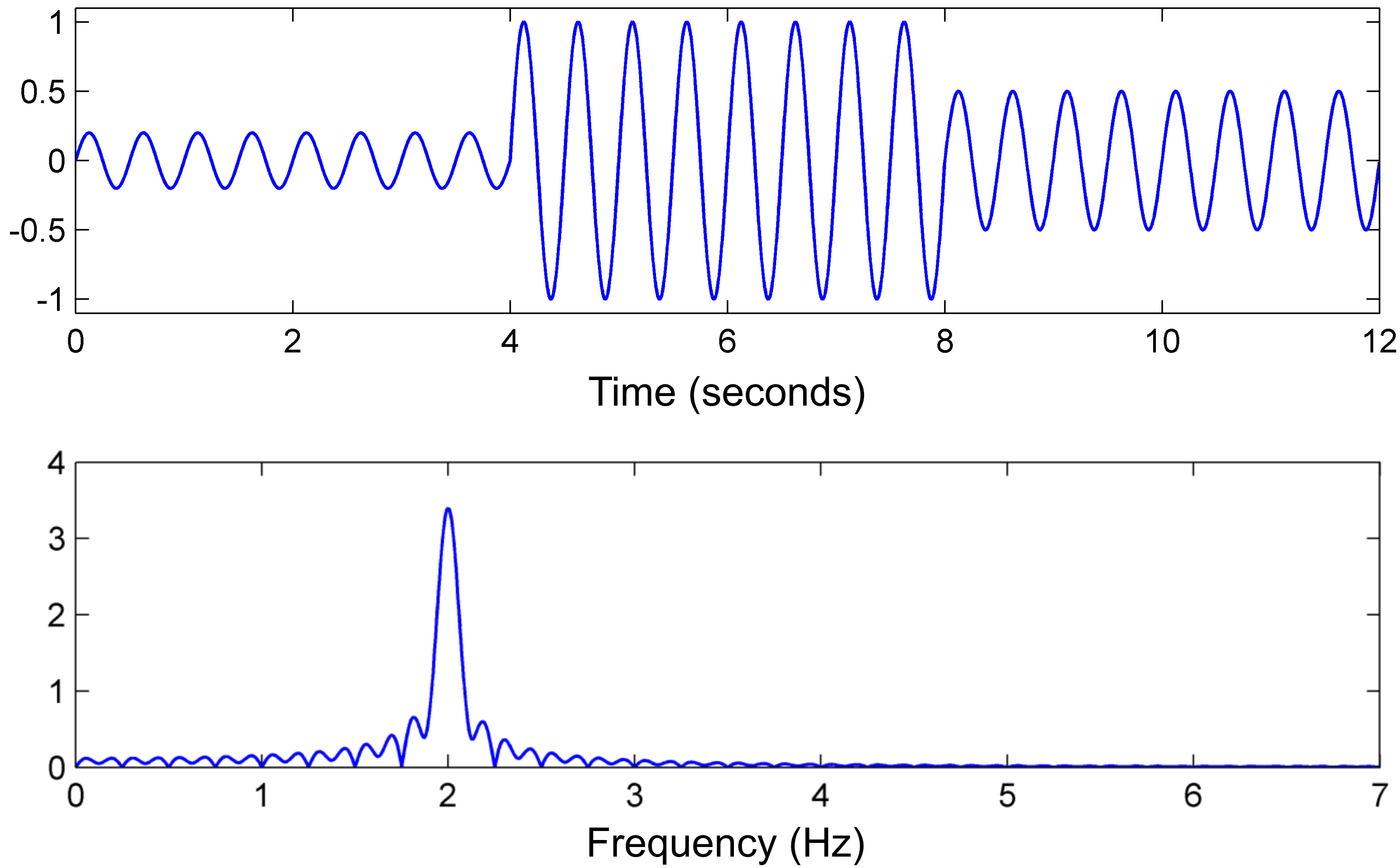
Homework
Exercise 2.2

Exercise 2.2. Sketch the magnitude Fourier transform of the following signals assuming that the signals are zero outside the shown intervals (see Figure 2.6 for similar examples):



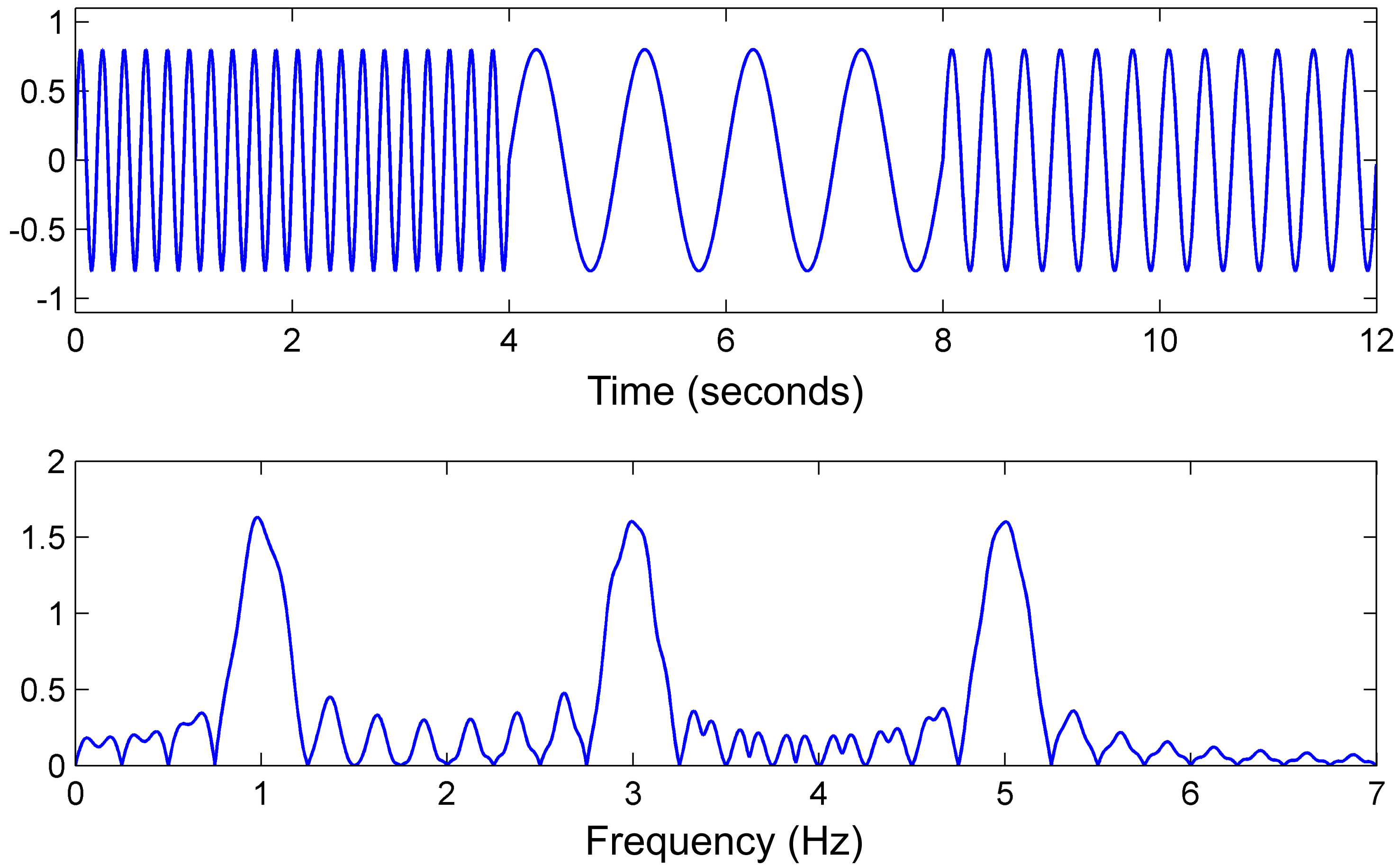
Homework

Solution 2.2a

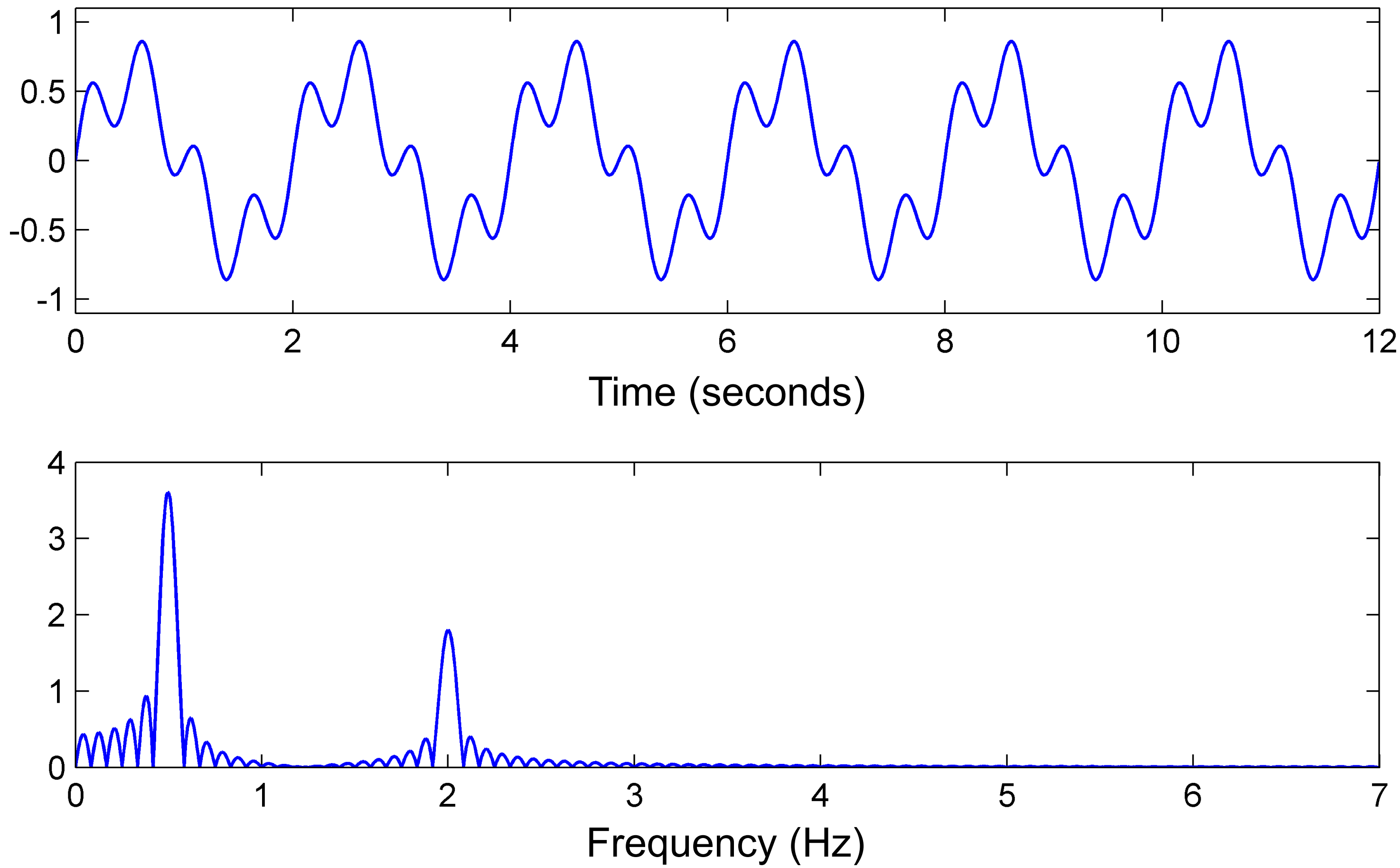


Homework

Solution 2.2b



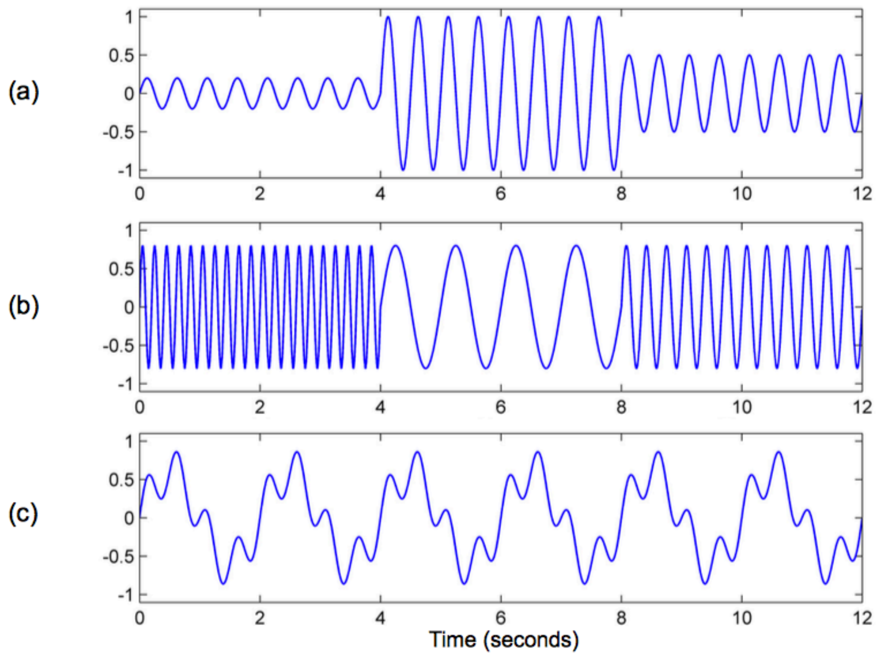
Homework
Solution 2.2c



Homework

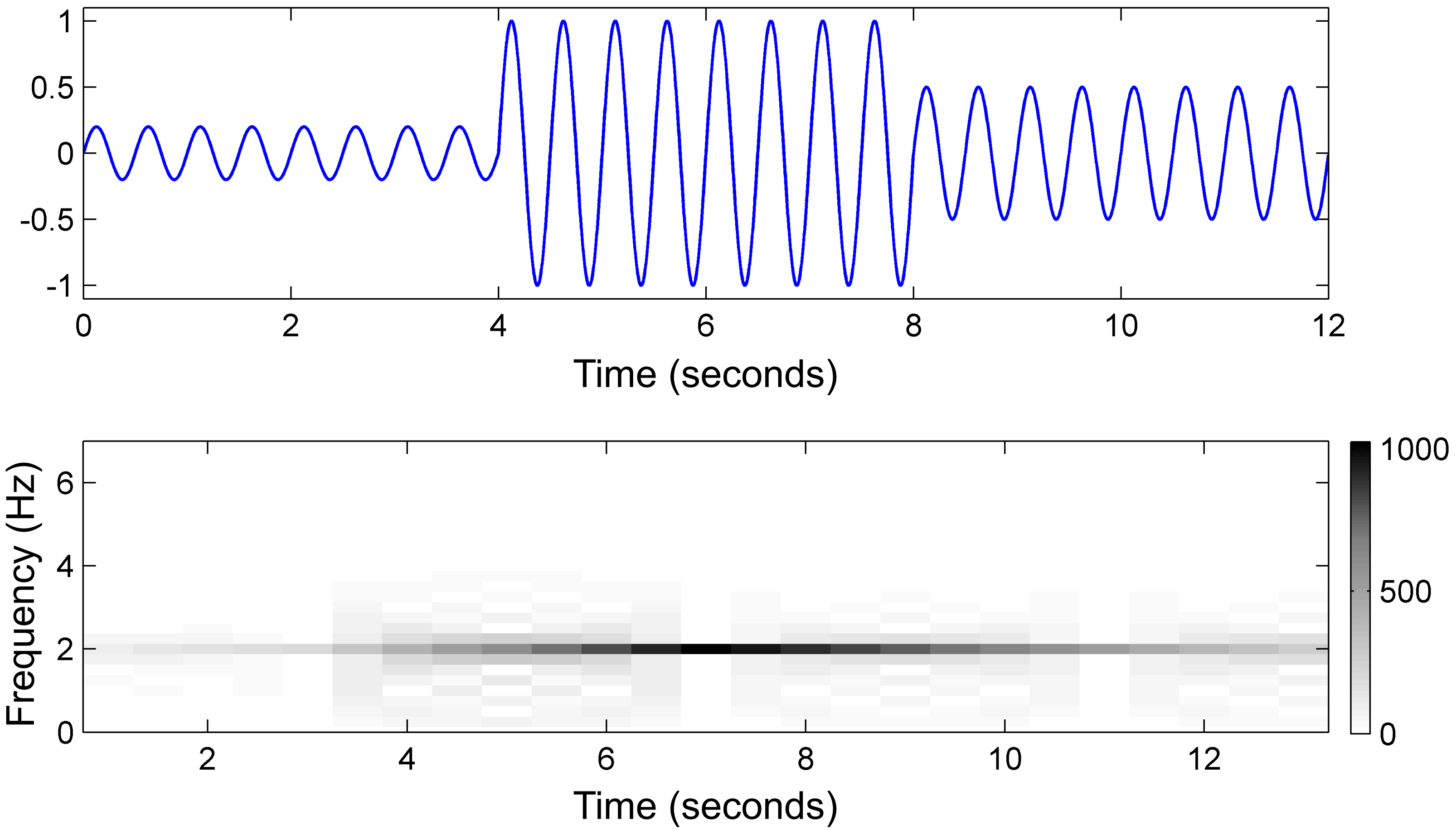
Exercise 2.5

Exercise 2.5. Sketch the magnitude Fourier transform (as in Figure 2.9) for each of the three signals shown in Exercise 2.2. Assume a window length that corresponds to a physical duration of about one second.

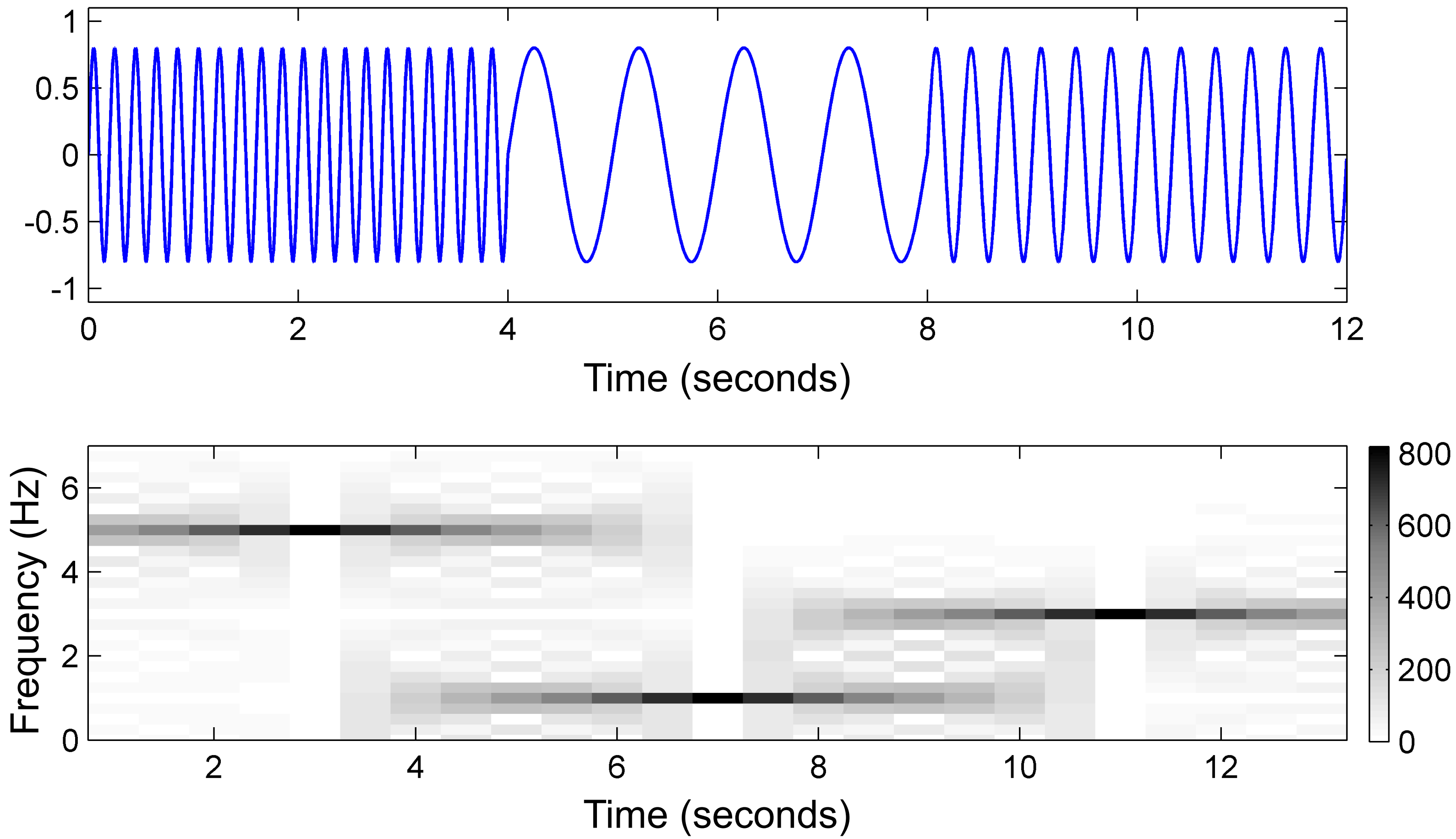


Homework

Solution 2.5a

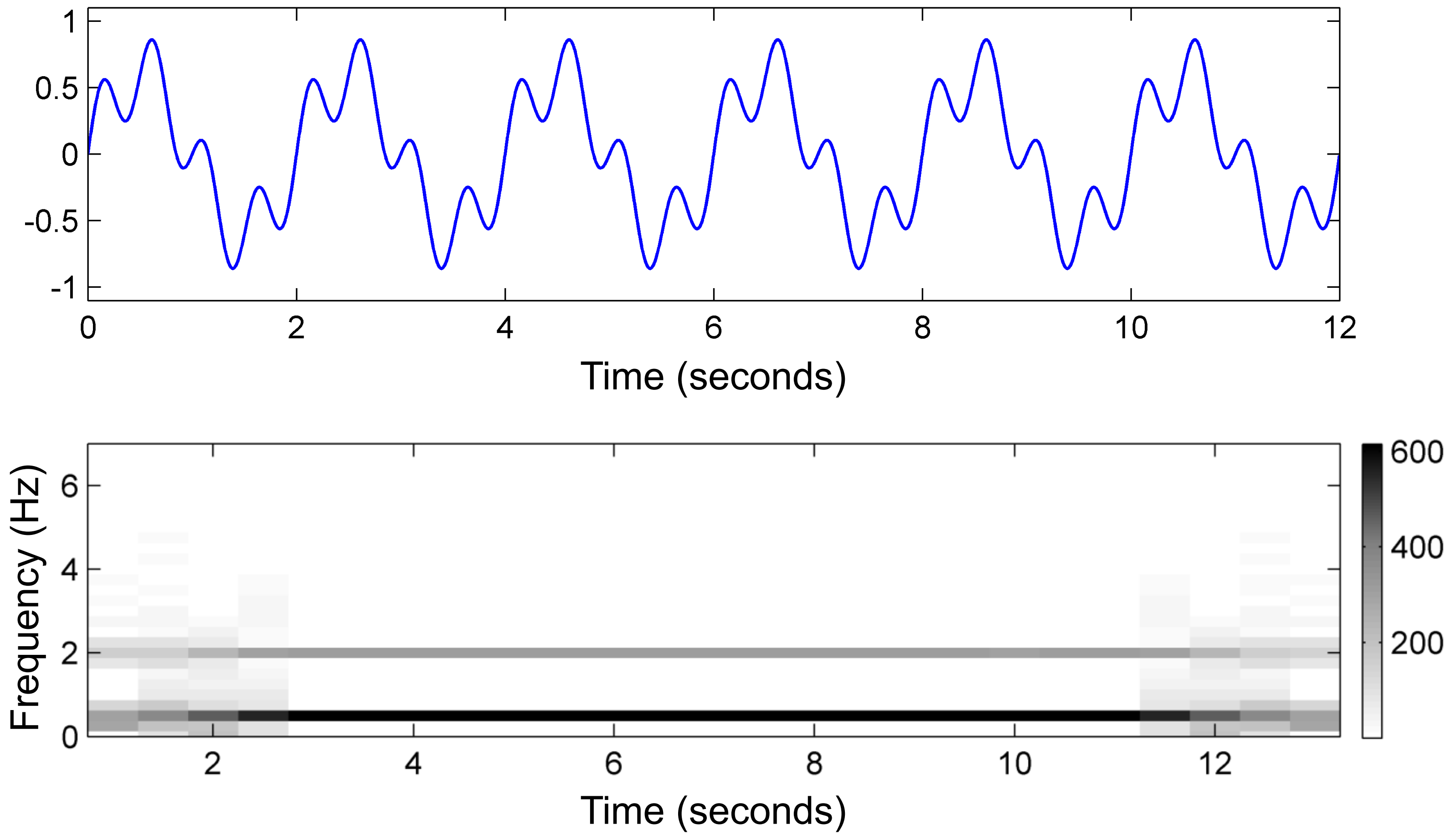


Homework
Solution 2.5b



Homework

Solution 2.5c



Homework
Exercise 2.12

Exercise 2.12. In this exercise we discuss various computation rules for complex numbers and their conjugates. The complex multiplication is defined by $c_1 \cdot c_2 = a_1a_2 - b_1b_2 + i(a_1b_2 + a_2b_1)$ for two complex numbers $c_1 = a_1 + ib_1, c_2 = a_2 + ib_2 \in \mathbb{C}$ (see (2.34)). Furthermore, complex conjugation is defined by $\overline{c} = a - ib$ for a complex number $c = a + ib \in \mathbb{C}$ (see (2.35)). Finally, the absolute value of a complex number c is defined by $|c| = \sqrt{a^2 + b^2}$. Prove the following identities:

(a) $\mathrm{Re}(c) = (c + \overline{c})/2$
(b) $\mathrm{Im}(c) = (c - \overline{c})/(2i)$
(c) $\overline{c_1 + c_2} = \overline{c_1} + \overline{c_2}$
(d) $\overline{c_1 \cdot c_2} = \overline{c_1} \cdot \overline{c_2}$
(e) $c\overline{c} = a^2 + b^2 = |c|^2$
(f) $1/c = \overline{c}/(c\overline{c}) = \overline{c}/(a^2 + b^2) = \overline{c}/(|c|^2)$

Homework
Solution 2.12

(a) Follows from $c + \overline{c} = a + ib + a - ib = 2a = 2\mathrm{Re}(c)$.
(b) Follows from $c - \overline{c} = a + ib - a + ib = 2ib = 2i\mathrm{Im}(c)$.
(c) $\overline{c_1 + c_2} = (a_1 + a_2) - i(b_1 + b_2) = (a_1 - ib_1) + (a_2 - ib_2) = \overline{c_1} + \overline{c_2}$
(d) $\overline{c_1 \cdot c_2} = a_1 a_2 - b_1 b_2 - i(a_1 b_2 + a_2 b_1) = (a_1 - ib_1)(a_2 - ib_2) = \overline{c_1} \cdot \overline{c_2}$
(e) $c\overline{c} = (a + ib)(a - ib) = a^2 + b^2 + i(-ab + ba) = a^2 + b^2 = |c|^2$
(f) Follows from $1 = c\overline{c}/(c\overline{c}) = c \cdot (\overline{c}/(c\overline{c}))$ and (e).

Homework

Exercise 2.14

Exercise 2.14. In Section 2.3.1, we defined the set $\{\mathbf{1}, \mathbf{sin}_k, \mathbf{cos}_k \mid k \in \mathbb{N}\} \subset L^2_{\mathbb{R}}([0,1))$. Prove that this set is an orthonormal set in $L^2_{\mathbb{R}}([0,1))$, i.e., that it satisfies (2.50) and (2.51).
[**Hint:** Use the following trigonometric identities:

(a) $\cos(\alpha)^2 + \sin(\alpha)^2 = 1$
(b) $\cos(\alpha)\cos(\beta) = (\cos(\alpha+\beta) + \cos(\alpha-\beta))/2$
(c) $\sin(\alpha)\sin(\beta) = (\cos(\alpha-\beta) - \cos(\alpha+\beta))/2$
(d) $\sin(\alpha)\cos(\beta) = (\sin(\alpha+\beta) + \sin(\alpha-\beta))/2$

To show (2.51), use (a) and the fact that $\mathbf{cos}_k^2$ and $\mathbf{sin}_k^2$ have the same area over a full period. The proof of (2.50) is a bit cumbersome, but not difficult when using (b), (c), and (d).]

$$\langle x_i | x_j \rangle = 0 \quad \text{for} \quad i, j \in I, i \neq j, \tag{2.50}$$

$$\|x_i\|^2 = 1 \quad \text{for} \quad i \in I, \tag{2.51}$$

Homework
Solution 2.14

Solution to Exercise 2.14. First, we prove (2.51). Obviously, one has $\|\mathbf{1}\|^2 = 1$. Furthermore, from identity (a), one obtains

$$2 = 2(\cos(2\pi kt)^2 + \sin(2\pi kt)^2) = \cos_k(t)^2 + \sin_k(t)^2 \quad (^*)$$

for all $t \in [0,1)$. Therefore,

$$2 = \int_{t\in[0,1)} \cos_k(t)^2 + \sin_k(t)^2 dt = \langle \cos_k | \cos_k \rangle + \langle \sin_k | \sin_k \rangle. \quad (^{**})$$

Both functions $\cos_k^2$ and $\sin_k^2$ are 1-periodic and shifted versions from each other. Therefore, integration of both functions over a full period yields the same value. As a result, one obtains $\langle \cos_k | \cos_k \rangle = \langle \sin_k | \sin_k \rangle = 1$.

(*)

$$\mathbf{cos}_k(t) := \sqrt{2}\cos(2\pi kt), \quad (2.55)$$

$$\mathbf{sin}_k(t) := \sqrt{2}\sin(2\pi kt), \quad (2.56)$$

(**)

$$\langle f | g \rangle := \int_{t\in[0,1)} f(t)\overline{g(t)}dt$$

Homework

Solution 2.14

Next, we prove (2.50).

$$\langle \mathbf{1} | \mathbf{cos}_k \rangle = \int_{t\in[0,1)} \sqrt{2}\cos(2\pi kt)dt = \left[\sqrt{2}\sin(2\pi kt)/(2\pi k)\right]_0^1 = 0$$

$$\langle \mathbf{1} | \mathbf{sin}_k \rangle = \int_{t\in[0,1)} \sqrt{2}\sin(2\pi kt)dt = \left[-\sqrt{2}\cos(2\pi kt)/(2\pi k)\right]_0^1 = 0$$

Using (b), one obtains for $k \neq \ell$:

$$\langle \mathbf{cos}_k | \mathbf{cos}_\ell \rangle = \int_{t\in[0,1)} \sqrt{2}\cos(2\pi kt)\sqrt{2}\cos(2\pi \ell t)dt$$

use (b)

$$= 2\int_{t\in[0,1)} \frac{\cos(2\pi(k+\ell)t) + \cos(2\pi(k-\ell)t)}{2} dt$$

$$= \left[\frac{\sin(2\pi(k+\ell)t)}{2\pi(k+\ell)} + \frac{\sin(2\pi(k-\ell)t)}{2\pi(k-\ell)}\right]_0^1 = 0$$

Homework
Solution 2.14

Similarly, using (c), one shows $\langle \mathbf{sin}_k | \mathbf{sin}_\ell \rangle = 0$ for $k \neq \ell$. For k ℓ, one obtains

$$\langle \mathbf{cos}_k | \mathbf{sin}_\ell \rangle = \int_{t\in[0,1)} \sqrt{2}\cos(2\pi kt)\sqrt{2}\sin(2\pi \ell t)dt$$

use (d)

$$= 2\int_{t\in[0,1)} \frac{\sin(2\pi(k+\ell)t) + \sin(2\pi(k-\ell)t)}{2}dt$$

$$=\text{-}\left[\frac{\cos(2\pi(k+\ell)t)}{2\pi(k+\ell)} + \frac{\cos(2\pi(k-\ell)t)}{2\pi(k-\ell)}\right]_0^1 = 0.$$

Finally, for $k = \ell$, one obtains

$$\langle \mathbf{cos}_k | \mathbf{sin}_k \rangle = \int_{t\in[0,1)} \sqrt{2}\cos(2\pi kt)\sqrt{2}\sin(2\pi kt)dt$$

$$= 2\int_{t\in[0,1)} \frac{\sin(2\pi(2k)t)}{2}dt = \left[\frac{-\cos(2\pi(2k)t)}{2\pi(k+\ell)}\right]_0^1 = 0.$$

This concludes the proof.